NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Extension 1 Mathematics Assessment Task 1
Term 4, 2012

Name:________________________________ **Mathematics Class: 11Mx**_______

Time Allowed: **50 minutes + 2 minutes reading time**

Total Marks: **47**

Instructions:

- Attempt all three questions.
- Start each question in a new booklet. Put your name on every booklet.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.

Question	1-3	4-5	6	7	8	Total
PE3	/3		/14		/8	/25
PE4		/2		/14		/16
HE7					/6	/6
	/3	/2	/14	/14	/14	/47

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Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5.

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1 Let α, β and γ be the roots of $P(x) = 2x^3 - 5x^2 + 4x - 9$.

Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(A) $\frac{5}{2}$ (B) $\frac{9}{2}$ (C) $\frac{4}{9}$ (D) $\frac{2}{5}$

2 Let $P(x) = 2x^3 - x^2 + 2$. Find the remainder when $P(x)$ is divided by $x + 1$.

(A) –1 (B) 1 (C) 3 (D) 5

3 Two roots of $4x^3 + 8x^2 - 9x + k = 0$ are equal in magnitude, but opposite in sign. Find the value of k.

(A) 2 (B) –2 (C) –46 (D) –18

4 The equation of the chord of contact to the parabola $x^2 = 8y$ from the point (3, –2) is

(A) $3x - 8y - 8 = 0$ (B) $3x - 4y + 16 = 0$

(C) $3x - 8y + 16 = 0$ (D) $3x - 4y + 8 = 0$

5 The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0 (B) 1 (C) 2 (D) 3

Section II

42 marks
Attempt Questions 6 – 8
Allow about 42 minutes for this section

Answer each question in a separate writing booklet.
In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (14 marks) Use a SEPARATE writing booklet

(a) Find the values of *A*, *B*, *C* and *D* in the identity **4**

$$2x^3 - 3 \equiv (x+3)(Ax^2 + Bx + C) + D$$

(b) The polynomial $p(x)$ is given by $p(x) = x^3 - 4x^2 + 4x - 3$

(i) Show that $x - 3$ is a factor of $p(x)$. **1**

(ii) Express $p(x)$ in the form $(x-3)(x^2 + bx + c)$, **2**
where *b* and *c* are integers.

(c) The cubic equation $3x^3 - x^2 - 38x - 24 = 0$ has roots such that one root is double the reciprocal of a second root.
By letting the roots be $\alpha, \dfrac{2}{\alpha}, \beta$ and considering the product of all the roots,
solve the equation. **3**

(d) Consider the polynomial $p(x)$, where $p(x) = x^3 - x^2 - 8x + 12$.

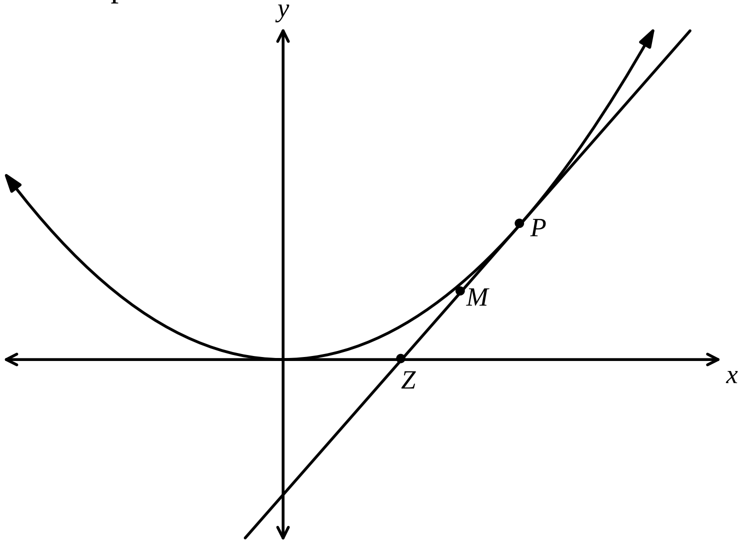
(i) Factorise $p(x)$ completely. **3**

(ii) The polynomial $q(x)$ has the form $q(x) = p(x)(x+a)$,
where *a* is a constant chosen so that $q(x) \geq 0$ for all real values of *x*.
Write down the value of *a*. **1**

Question 7 (14 marks) Use a SEPARATE writing booklet

(a) The tangent at $P(4t, 2t^2)$ to the parabola $x^2 = 8y$ intersects the x-axis at Z.

The point M is the midpoint of PZ.



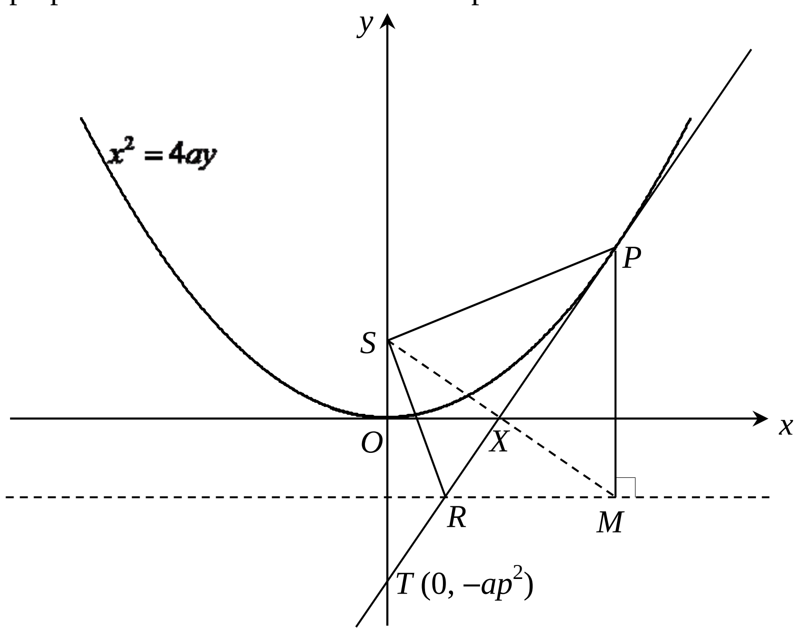
(i) Show that the equation of the tangent is $y = tx - 2t^2$. **2**

(ii) Show that the M has coordinates $(3t, t^2)$ **2**

(iii) Find the Cartesian equation of the locus of M as t varies. **2**

(b) In the diagram below, P is the point $(2ap, ap^2)$. PT is the tangent to the parabola at P.

PM is perpendicular to the directrix of the parabola.



R and X are the intersections of the tangent with the directrix and x-axis respectively.

S is the focus.

The equation of the tangent is given by $y = px - ap^2$. (Do NOT prove this)

(i) Prove that $MPST$ is a rhombus. **3**

(ii) Prove that $\Delta RSP \equiv \Delta RMP$ **3**

(iii) Hence prove that $RMPS$ is a cyclic quadrilateral. **2**

Question 8 (14 marks) Use a SEPARATE writing booklet

(a) Consider the curve $y = \dfrac{(x-1)^2}{x^2+1}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2(x^2-1)}{(x^2+1)^2}$ **2**

(ii) Find the coordinates of the stationary points and determine the nature of the curve's stationary points. **3**

(iii) Write down the equation of the horizontal asymptote. **1**

(iv) Sketch the curve, showing all important features. **2**
You are NOT required to find any points of inflexion.

(b) A curve is defined by the parametric equations $x = t + \dfrac{1}{t}$ and $y = t^2 + \dfrac{1}{t^2}$ for $t \neq 0$.

(i) Find the Cartesian equation of the curve. **2**

(ii) By considering the discriminant, or otherwise, find the values of *k* for which $x = k$ has solutions, where *k* is a constant. **2**

(iii) Sketch the curve, showing any domain restrictions implied by the above parts **2**

End of paper

NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Extension 1 Mathematics Assessment Task 1
Term 4, 2012

Sample Solutions

MC Answers

1. C
2. A
3. D
4. D
5. B

Section I

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1 Let α, β and γ be the roots of $P(x) = 2x^3 - 5x^2 + 4x - 9$.

Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(A) $\frac{5}{2}$ (B) $\frac{9}{2}$ **(C)** $\frac{4}{9}$ (D) $\frac{2}{5}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{\frac{4}{2}}{\frac{9}{2}} = \frac{4}{9}$$

2 Let $P(x) = 2x^3 - x^2 + 2$. Find the remainder when $P(x)$ is divided by $x + 1$.

(A) –1 (B) 1 (C) 3 (D) 5

$P(-1) = 2(-1)^3 - (-1)^2 + 2 = -1$

So the remainder is –1.

3 Two roots of $4x^3 + 8x^2 - 9x + k = 0$ are equal in magnitude, but opposite in sign. Find the value of k.

(A) 2 (B) –2 (C) –46 **(D)** –18

Let the roots be α, $-\alpha$ and β.

$\therefore \alpha + (-\alpha) + \beta = -\frac{8}{4} = -2$

$\therefore \beta = -2$

$\therefore x = -2$ is a root of the equation

Substitute $x = -2$ into the equation: $4(-2)^3 + 8(-2)^2 - 9(-2) + k = 0$

$\therefore -32 + 32 + 18 + k = 0 \Rightarrow k = -18$.

4 The equation of the chord of contact to the parabola $x^2 = 8y$ from the point (3, –2) is

(A) $3x - 8y - 8 = 0$ (B) $3x - 4y + 16 = 0$

(C) $3x - 8y + 16 = 0$ **(D)** $3x - 4y + 8 = 0$

The chord of contact is $xx_0 = 2a(y + y_0)$; $a = 2$; $(x_0, y_0) = (3, -2)$.

$\therefore 3x = 4(y - 2)$ is the chord of contact i.e. $3x - 4y + 8 = 0$.

5 The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0 **(B)** 1 (C) 2 (D) 3

Only the normal at (0, 0) passes through the focus.

OR

Substitute (0, a) into the equation of the normal i.e. $pa = 2ap + ap^3$

$\therefore ap^3 + ap = 0 \Rightarrow ap(p^2 + 1) = 0$

$\therefore p = 0$ i.e. (0, 0) is the only point where the normal passes through the focus.

Section II

Question 6 (14 marks)

(a) Find the values of *A*, *B*, *C* and *D* in the identity $2x^3 - 3 \equiv (x+3)(Ax^2 + Bx + C) + D$ **4**

$A = 2$ (comparing coefficients of x^2.)

Substitute $x = -3 \Rightarrow 2(-3)^3 - 3 = D$

$\therefore D = -57.$

Substitute $x = 0 \Rightarrow -3 = 3C - 57$

$\therefore 3C = 54 \Rightarrow C = 18$.

Substitute $x = 1 \Rightarrow 2 - 3 = 4(2 + B + 18) - 57$

$\therefore B + 20 = 14$

$\therefore B = -6$

$\therefore A = 2, B = -6, C = 18, D = -57$

(b) The polynomial $p(x)$ is given by $p(x) = x^3 - 4x^2 + 4x - 3$

(i) Show that $x - 3$ is a factor of $p(x)$. **1**

Find $p(3)$: $p(3) = 3^3 - 4 \times 3^2 + 4 \times 3 - 3 = 0$

$\therefore x - 3$ is a factor of $p(x)$.

(ii) Express $p(x)$ in the form $(x-3)(x^2 + bx + c)$, **2**

where *b* and *c* are integers.

From (i): $p(x) = (x-3)(x^2 + Bx + 1),\ B \in \mathbb{R}$

Find *p*(1): $p(1) = 1 - 4 + 4 - 3 = -2$

$\therefore p(1) = -2 = (1-3)(2+B)$

$\therefore 2 + B = 1 \Rightarrow B = -1$

$\therefore p(x) = (x-3)(x^2 - x + 1)$

(c) The cubic equation $3x^3 - x^2 - 38x - 24 = 0$ has roots such that one root is double the reciprocal of a second root. By letting the roots be $\alpha, \dfrac{2}{\alpha}, \beta$ and considering the product of all the roots, solve the equation. **3**

$\alpha \times \frac{2}{\alpha} \times \beta = -\frac{-24}{3} = 8$

$\therefore 2\beta = 8 \Rightarrow \beta = 4$

$\therefore x - 4$ is a factor of $3x^3 - x^2 - 38x - 24 = 0$.

$\therefore 3x^3 - x^2 - 38x - 24 = (x-4)(3x^2 + Bx + 6),\ B \in \mathbb{R}$

Substitute $x = 1$: $3 - 1 - 38 - 24 = (-3)(9 + B)$

$\therefore 9 + B = 20$

$\therefore B = 11$

$\therefore 3x^3 - x^2 - 38x - 24 = (x-4)(3x^2 + 11x + 6)$

$\therefore 3x^3 - x^2 - 38x - 24 = (x-4)(3x+2)(x+3)$

$\therefore 3x^3 - x^2 - 38x - 24 = 0 \Rightarrow x = -3, -\frac{2}{3}, 4$

(d) Consider the polynomial $p(x)$, where $p(x)=x^3-x^2-8x+12$.

(i) Factorise $p(x)$ completely. **3**

Test $x=\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$p(1)=1-1-8+12=4$

$p(-1)=-1-1+8+12=18$

$p(2)=8-4-16+12=0$

$\therefore x-2$ is a factor.

$\therefore p(x)=x^3-x^2-8x+12=(x-2)(x^2+Bx-6), B\in\mathbb{R}$.

Consider $p(1)=4$:

$$p(1)=4 \Rightarrow 4=(-1)(B-5)$$
$$\therefore B-5=-4 \Rightarrow B=1$$

$$\begin{aligned}\therefore p(x)&=x^3-x^2-8x+12\\&=(x-2)(x^2+x-6)\\&=(x-2)^2(x+3)\end{aligned}$$

(ii) The polynomial $q(x)$ has the form $q(x)=p(x)(x+a)$, where a is a constant chosen so that $q(x)\geq 0$ for all real values of x. Write down the value of a. **1**

$q(x)=(x+a)\,p(x)\geq 0$

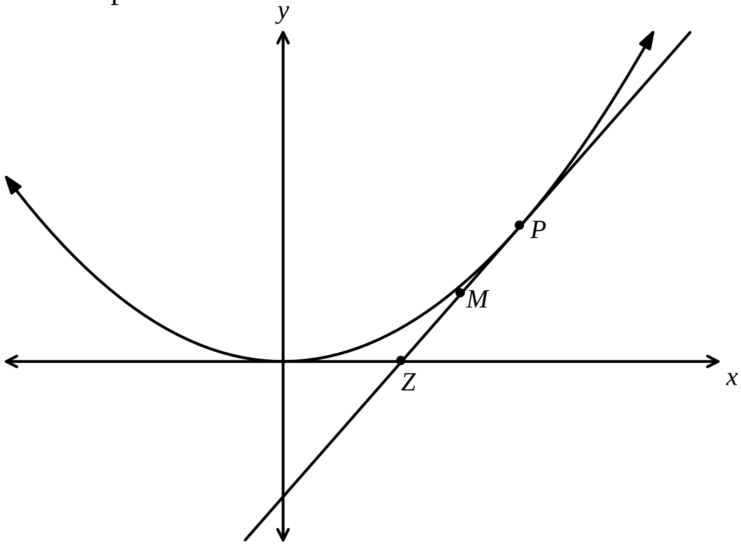
$\therefore q(x)=(x-2)^2(x+3)^2$

$\therefore a=3$

Question 7 (14 marks)

(a) The tangent at $P\left(4t, 2t^2\right)$ to the parabola $x^2 = 8y$ intersects the x-axis at Z.

The point M is the midpoint of PZ.



(i) Show that the equation of the tangent is $y = tx - 2t^2$. **2**

Parametric	Cartesian
$x = 4t \Rightarrow \frac{dx}{dt} = 4$	$y = \frac{1}{8}x^2$
$y = 2t^2 \Rightarrow \frac{dy}{dt} = 4t$	$\therefore y' = \frac{1}{4}x$
$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t}{4} = t$	At $P\left(4t, 2t^2\right)$, $y' = \frac{1}{4} \times 4t = t$

$\therefore y - 2t^2 = t\left(x - 4t\right)$

$\therefore y = tx - 2t^2$

(ii) Show that the M has coordinates $\left(3t, t^2\right)$ **2**

Substitute y = 0: Z has coordinates $\left(2t, 0\right)$

M is the midpoint of $Z\left(2t, 0\right)$ and $P\left(4t, 2t^2\right)$.

$\therefore M$ has coordinates $\left(\frac{2t + 4t}{2}, \frac{0 + 2t^2}{2}\right) = \left(3t, t^2\right)$.

(iii) Find the Cartesian equation of the locus of M as t varies. **2**

Let $x = 3t$ and $y = t^2$.

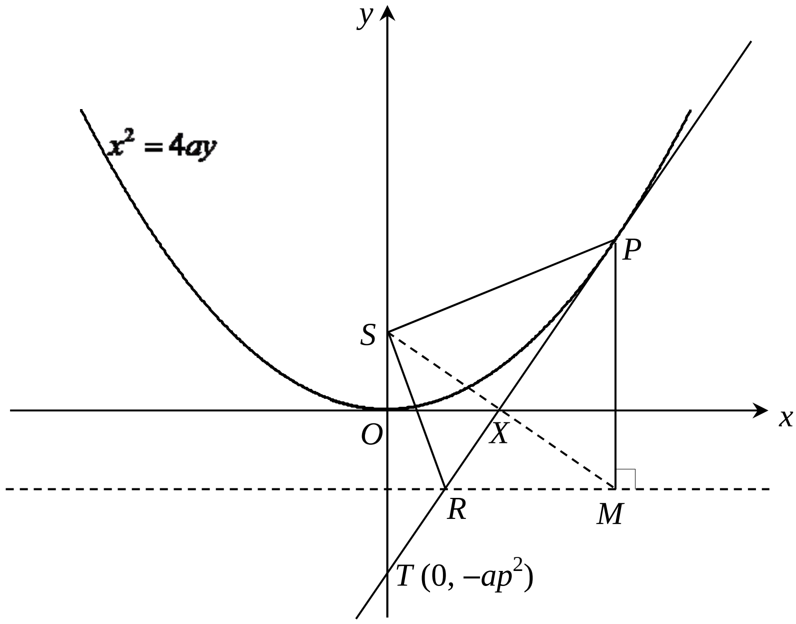
$x = 3t \Rightarrow t = \frac{x}{3}$

$\therefore y = \left(\frac{x}{3}\right)^2$

$\therefore x^2 = 9y$

So the locus of M is $x^2 = 9y$.

(b) In the diagram below, P is the point $\left(2ap, ap^2\right)$. PT is the tangent to the parabola at P. PM is perpendicular to the directrix of the parabola.



R and X are the intersections of the tangent with the directrix and x-axis, respectively. S is the focus.
The equation of the tangent is given by $y = px - ap^2$. (Do NOT prove this)

(i) Prove that $MPST$ is a rhombus. **3**

M has coordinates $\left(2ap, -a\right)$.

So using the diagram, $ST = a + ap^2 = PM$.

Using the locus definition of a parabola, $SP = PM$.

$\therefore ST = PM = SP = a + ap^2$.

$$TM^2 = \left(0 - 2ap\right)^2 + \left(-ap^2 + a\right)^2$$
$$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$$
$$= a^2p^4 + 2a^2p^2 + a^2$$
$$= \left(ap^2 + a\right)^2$$

$\therefore ST = PM = SP = TM = a + ap^2$.

> **Alternative:**
> Prove that PT & SM are perpendicular
> Also prove that SM <u>and</u> PT have the SAME midpoint.

(ii) Prove that $\Delta RSP \equiv \Delta RMP$ **3**

$SP = PM$ (shown above)

$\angle SPR = \angle MPR$ (diagonals of rhombus)

PR is common.

$\therefore \Delta RSP \equiv \Delta RMP$ (SAS).

(iii) Hence prove that $RMPS$ is a cyclic quadrilateral. **2**

$\angle PSR = \angle PMR$ (matching sides cong. Δs)

$= 90^\circ$ (given)

Question 8 (14 marks)

(a) Consider the curve $y = \dfrac{(x-1)^2}{x^2+1}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2(x^2-1)}{(x^2+1)^2}$ **2**

$$\frac{dy}{dx} = \frac{(x^2+1)\times 2(x-1)-(x-1)^2\times 2x}{(x^2+1)^2} = \frac{2(x-1)\left[(x^2+1)-x(x-1)\right]}{(x^2+1)^2}$$

$$= \frac{2(x-1)(1+x)}{(x^2+1)^2} = \frac{2(x^2-1)}{(x^2+1)^2}$$

(ii) Find the coordinates of the stationary points and determine the nature of the curve's stationary points. **3**

Stationary points when $y' = 0$ i.e. $2(x^2-1) = 0$.

$\therefore x = \pm 1$

Stationary points are at (1, 0) and (–1, 2)

NB Test only the numerator as $(x^2+1)^2 > 0$

x	–2	–1	0	1	2
y'	6	0	–2	0	6
	/	–	\	–	/

(1, 0) is a rel. min. turning point and (–1, 2) is a rel. max. turning point.

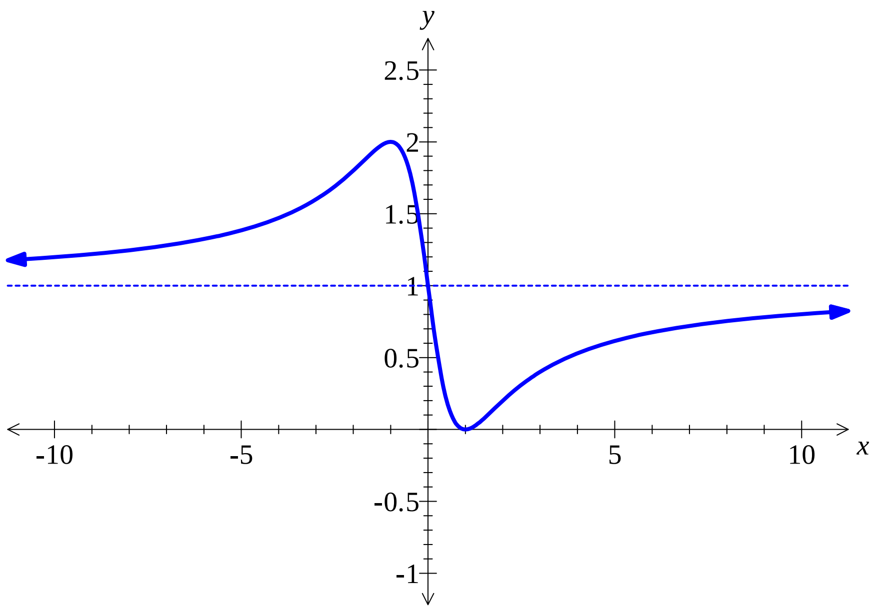
(iii) Write down the equation of the horizontal asymptote. **1**

$y = 1$. Why? $\lim_{x\to\infty} \dfrac{(x-1)^2}{x^2+1} = 1$

(iv) Sketch the curve, showing all important features. **2**

You are NOT required to find any points of inflexion.

x-int: $x = 0$; y-int: $y = 1$; No symmetry



(b) A curve is defined by the parametric equations $x = t + \frac{1}{t}$ and $y = t^2 + \frac{1}{t^2}$ for $t \neq 0$.

(i) Find the Cartesian equation of the curve. **2**

$$y = t^2 + \frac{1}{t^2}$$
$$= \left(t + \frac{1}{t}\right)^2 - 2$$
$$= x^2 - 2$$

(ii) By considering the discriminant, or otherwise, find the values of k for which $x = k$ has solutions, where k is a constant. **2**

<u>Considering the discriminant</u>

$$x = t + \frac{1}{t} = k$$
$$\therefore t^2 - tk + 1 = 0$$
$$\therefore \Delta = k^2 - 4$$

To have solutions, $\Delta \geq 0$.

$$\therefore k^2 - 4 \geq 0$$
$$\therefore (k-2)(k+2) \geq 0$$
$$\therefore k \leq -2, k \geq 2$$

<u>Or otherwise</u>

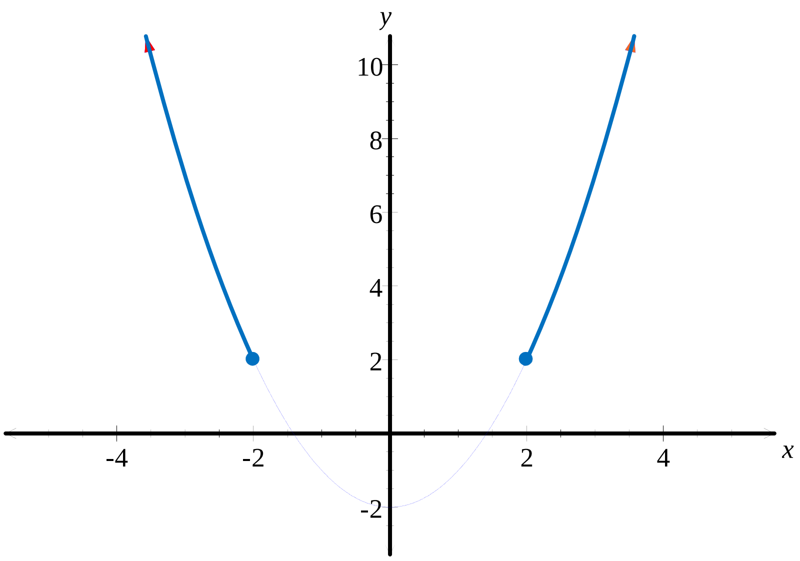
For $t > 0$, $t + \frac{1}{t} \geq 2$ and so if $t + \frac{1}{t} = k$ then $k \geq 2$.

Similarly, if $t < 0$ then $t + \frac{1}{t} \leq -2$ and so $k \leq -2$

$$\therefore k \leq -2, k \geq 2$$

(iii) Sketch the curve, showing any domain restrictions implied by the above parts **2**

Part (ii) gives the domain restrictions on the locus i.e. $x \leq -2, x \geq 2$ as the values of k are the possible values that x can take.



End of solutions